AP Calculus AB — Homework 0212 Solutions

Dr. Paul L. Bailey — Wednesday, February 12, 2025

Problem 1 (Thomas §4.2 # 4)**.** Let $f(x) = \sqrt{x-1}$. Let $a = 1$ and $b = 3$. Find $c \in [a,b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Solution. First compute $f'(x) = \dfrac{1}{2\sqrt{x-1}}$. Then compute $\dfrac{f(b)-f(a)}{b-a} = \dfrac{f(3)-f(1)}{3-1} = \dfrac{\sqrt{2}}{2}$. Set $\dfrac{1}{2\sqrt{x-1}} = \dfrac{\sqrt{2}}{2}$ and solve for x. Thus $c = \dfrac{3}{2}$. □

Problem 2 (Thomas §4.2 # 10)**.** Let

$$f(x) = \begin{cases} 3 & \text{for } x = 0 \\ -x^2 + 3x + a & \text{for } x \in (0,1) \\ mx + b & \text{for } x \in [1,2] \end{cases}$$

For what values of a, m, and b does f satisfy the hypothesis of the Mean Value Theorem on the interval $[0,2]$?

Solution. The function must be differentiable at $x = 1$, so $-2x + 3 = m$ at $x = 1$, so $m = -2 + 3 = 1$. The function must be continuous at $x = 0$, so $3 = a$. The function must be continuous at $x = 1$, so $-x^2 + 3x + 3 = x + b$ when $x = 1$, so $5 = 1 + b$, so $b = 4$. □

Problem 3 (Thomas §4.2 # 15)**.** Show that the function

$$f(x) = x^4 + 3x + 1$$

has exactly one zero on $[-2,-1]$.

Solution. Note that $f'(x) = 4x^3 + 3$. A sign chart for f' tells us that f is decreasing for $x < -\sqrt[3]{\frac{3}{4}}$; thus f is injective on $[-2,-1]$. Now $f(-2) = 11$ and $f(-1) = -1$, so there exists $c \in (-2,-1)$ such that $f(c) = 0$ by the Intermediate Value Theorem, and it is unique by injectivity. □

Problem 4 (Thomas §4.2 # 19)**.** Show that the function

$$r(\theta) = \theta + \sin^2(\theta/3) - 8$$

has exactly one zero on $\mathbb{R}$.

Solution. Note that $\dfrac{dr}{d\theta} = 1 + \frac{2}{3}\sin(\theta/3)\cos(\theta/3)$. Since $|\sin(\theta/3)\cos(\theta/3)| \leq 1$, this is always positive, so r is increasing, and thus injective. Moreover, $r(0) < 0$ and $r(8) > 0$, so r has a zero by IVT. □

Problem 5 (Thomas §3.7 # 27)**.** A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

Solution. This is a related rates problem. Follow these steps.

- Draw it.
- Label variables and write down relations (equations).
- Identify the cheese. This is typically of the form $\frac{dz}{dt}$, where z is one of your variables.
- Take $\frac{d}{dt}$ of both sides of the main equation that has z in it. Solve for $\frac{dz}{dt}$.

Our main relations and cheese are:

- $y = x^2$
- $\frac{dx}{dt} = 10$
- $\tan\theta = \frac{y}{x}$ and $\sec\theta = \frac{\sqrt{x^2 + y^2}}{x}$
- Cheese: $\frac{d\theta}{dt}$ when $x = 3$

Take $\frac{d}{dt}$ of both sides of $\tan\theta = \frac{y}{x}$ to get

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - y\frac{dx}{dt}}{x^2},$$

so

$$\frac{d\theta}{dt} = \frac{\frac{dy}{dt}x - y\frac{dx}{dt}}{x^2\sec^2(\theta)} = \frac{\frac{dy}{dt}x - y\frac{dx}{dt}}{x^2 + y^2}.$$

Note that when $x = 3$, we have $\frac{dy}{dt} = 2x\frac{dx}{dt} = 60$. Set $x = 3$ so $y = 9$ and plug in to get

$$\frac{d\theta}{dt} = \frac{(60)(3) - (9)(10)}{9 + 81} = 1.$$

□

Problem 6 (Thomas §3.6 # 46)**.** Consider the equation

$$(x^2 + y^2)^2 = (x - y)^2.$$

Find the slope of the curve at $(1, 0)$ and $(1, -1)$.

Solution. Implicitly differentiate the equation to get

$$2(x^2 + y^2)(2x + 2yy') = 2(x - y)(1 - y').$$

□

Problem 7 (Thomas §4.1 #4)**.** Let

$$f(x) = \frac{x+1}{x^2+2x+2}.$$

Find all local extreme values of the function f, and where they occur.

Solution. Let $g(x) = \dfrac{x}{x^2+1}$. We previously found that $g(x)$ has a local max at $(1, 1/2)$ and a local min at $(-1, -1/2)$. Now $f(x) = g(x+1)$, so its graph is the graph of g shifted left by 1. Thus its local min is $(0, 1/2)$ and $(-2, -1/2)$. □

Problem 8. Let

$$f(x) = x^3 - 7x + 6.$$

Let $a, b, c \in \mathbb{R}$ with $a < b < c$ and $f(a) = f(b) = f(c)$. Let $A = [a, c]$ and $B = f(A)$. Write B in interval notation.

Solution. Factor $f(x) = (x+3)(x-1)(x-2)$. So $a = -3$, $b = 1$, $c = 2$. Compute $f'(x) = 3x^2 - 7$, so $f'(x) = 0$ implies that $x = \pm\sqrt{\frac{7}{3}}$. Let $a = \sqrt{\frac{7}{3}}$. The range is $[f(-a), f(a)] = [6 - \sqrt{\frac{7}{3}}, 6 + \sqrt{\frac{7}{3}}]$. □

Problem 9. Consider the polynomial

$$f(x) = x^4 - 2x^2 - 15.$$

Find all real zeros of the f. (Hint: Factor by Substitution $u = x^2$)

Solution. We have

$$f(x) = x^4 - 2^2 - 15 = (x^2+3)(x^2-5) = (x - \sqrt{3}i)(x + \sqrt{3}i)(x - \sqrt{5})(x + \sqrt{5}).$$

The *real* zeros are $\pm\sqrt{5}$. □

Problem 10. Consider the polynomial

$$f(x) = 3x^3 + 11x^2 - 19x + 5.$$

Find all real zeros of the f. (Hint: Rational Zeros Theorem)

Solution. By the Rational Zeroes Theorem, the only possible rational zeros are

$$\pm 1, \pm 5, \pm\frac{1}{3}, \pm\frac{5}{3}.$$

We try these one are a time starting at the easiest. Now $f(1) = 3 + 11 - 19 + 5 = 0$. By the Factor Theorem, $f(x)$ is divisible by $x - 1$. Synthetic division gives

$$f(x) = (x-1)(3x^2 + 14x - 5) = (x-1)(3x-1)(x+5).$$

So the real zeros are 1, $\frac{1}{3}$, and -5. □